

RELATIVE BGG SEQUENCES; I. ALGEBRA

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ABSTRACT. We develop a relative version of Kostant's harmonic theory and use this to prove a relative version of Kostant's theorem on Lie algebra (co)homology. These are associated to two nested parabolic subalgebras in a semisimple Lie algebra. We show how relative homology groups can be used to realize representations with lowest weight in one (regular or singular) affine Weyl orbit. In the regular case, we show how all the weights in the orbit can be realized as relative homology groups (with different coefficients). These results are motivated by applications to differential geometry and the construction of invariant differential operators.

1. INTRODUCTION

This article is the first in a series of two. The main aim of the series is to develop a relative version of the machinery of Bernstein–Gelfand–Gelfand sequences (or BGG sequences) as introduced in [6] and [3] and to improve the original constructions at the same time, which is done in [7]. This is a construction for invariant differential operators associated to a class of geometric structures known as parabolic geometries. For each type of parabolic geometries, there is a homogeneous model, which is a generalized flag manifold, i.e. the quotient of a (real or complex) semisimple Lie group G by a parabolic subgroup P . The starting point for the construction of a BGG sequence is a finite-dimensional representation \mathbb{V} of G . On the homogeneous model, the resulting sequence is a resolution of the locally constant sheaf \mathbb{V} on G/P by differential operators acting on spaces of sections of homogeneous vector bundles induced by irreducible representations of P . The resulting resolution of \mathbb{V} by principal series representations of G is dual (in a certain sense) to the resolution of \mathbb{V}^* by generalized Verma-modules obtained in [11]. This generalizes the Bernstein–Gelfand–Gelfand resolution of \mathbb{V}^* by Verma-modules from [1], which motivated the name of the construction.

The algebraic character of BGG sequences is also reflected by the tools needed for their construction, and this first part of the series is devoted to developing the necessary algebraic background for the relative version. In particular, we prove a relative version of Kostant's theorem on Lie algebra cohomology from [9], which should be of independent interest. The setup for Kostant's original theorem is a complex semisimple Lie algebra \mathfrak{g} , a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ with (reductive) Levi-decomposition $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{p}_+$ and a complex irreducible representation \mathbb{V} of \mathfrak{g} .

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Then Kostant considered the standard complex $(C^*(\mathfrak{p}_+, \mathbb{V}), \partial)$ computing the Lie algebra cohomology of the nilpotent Lie algebra \mathfrak{p}_+ with coefficients in \mathbb{V} . The spaces in this complex are naturally representations of \mathfrak{g}_0 and the differentials are \mathfrak{g}_0 -equivariant. Thus the cohomology groups $H^*(\mathfrak{p}_+, \mathbb{V})$ are representations of the reductive Lie algebra \mathfrak{g}_0 and Kostant's theorem describes these representations explicitly and algorithmically in terms of highest weights.

While higher Lie algebra cohomology groups seem to be difficult to interpret in general, Kostant's theorem has immediate algebraic applications, see [9]. Even the version for the Borel subalgebra (which in some respects is significantly simpler than the general result) very quickly implies the Weyl character formula, thus providing a completely algebraic proof for this formula. Moreover, together with the Peter–Weyl theorem, Kostant's theorem can be used to prove Bott's generalized version (see [2]) of the Borel–Weil theorem describing the sheaf cohomology of the sheaf of local holomorphic sections of a homogeneous vector bundle over a complex generalized flag manifold. Apart from the applications in the theory of parabolic geometries (see [5]), Kostant's theorem has also been applied in other areas recently. For example, in [10], Lie algebra cohomology as computed via Kostant's theorem is used as a replacement for Spencer cohomology in connection with exterior differential systems to prove rigidity results in algebraic geometry.

It is important to point out here that for the applications to parabolic geometries, not only Kostant's theorem itself is important. Also some of the tools introduced by Kostant in the proof play a central role there. These tools are also available for parabolic subalgebras in real semisimple Lie algebras, and the real versions are needed in the applications. Moreover, for these applications it is very important to carefully keep track about the possibility of formulating things in a \mathfrak{p} -invariant way. Likewise, one has to carefully distinguish between filtered modules and their associated graded modules in this setting, while this distinction plays no role for \mathfrak{g}_0 -modules. In view of the applications in [7], we will work in a real setting for most of the article and be more careful about invariance under the parabolic subalgebra than it would be necessary for the purposes of the current article.

The setup for the relative version is provided by two nested parabolic subalgebras $\mathfrak{q} \subset \mathfrak{p}$ in a semisimple Lie algebra \mathfrak{g} . Here \mathfrak{q} will be the “main” parabolic subalgebra (so \mathfrak{q} -invariance will be important), while the larger parabolic \mathfrak{p} is an auxiliary input. If \mathbb{V} is an irreducible representation of \mathfrak{p} , then the nilradical $\mathfrak{p}_+ \subset \mathfrak{p}$ acts trivially on \mathbb{V} . Moreover, the nilradicals satisfy $\mathfrak{p}_+ \subset \mathfrak{q}_+ \subset \mathfrak{p}$, so we can naturally view \mathbb{V} as a representation of the nilpotent Lie algebra $\mathfrak{q}_+/\mathfrak{p}_+$. In view of \mathfrak{q} -invariance, it is preferable to work with Lie algebra homology rather than Lie algebra cohomology. The standard complex for Lie algebra homology consists of \mathfrak{q} -modules and \mathfrak{q} -equivariant maps, so the Lie algebra homology groups $H_*(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V})$ are automatically representations of \mathfrak{q} . These are completely reducible, and hence can (in the complex case) be described via weights. This description is the content of the relative version of Kostant's theorem, which we prove as Theorem 2.7. As a module for the Levi-factor of \mathfrak{q} , one can identify this with Lie algebra cohomology for the quotients of the nilradicals of the opposite

parabolics. As in Kostant's original theorem, the description is in terms of the orbit of an extremal weight of \mathbb{V} under a subset $W_{\mathfrak{p}}^{\mathfrak{q}}$ of the Weyl group of \mathfrak{g} .

The second main topic of the article is the relation between absolute and relative homology groups in the case of regular infinitesimal character. Suppose that, given $\mathfrak{q} \subset \mathfrak{p} \subset \mathfrak{g}$, we take an irreducible representation $\tilde{\mathbb{V}}$ of \mathfrak{g} and let \mathbb{V} be its \mathfrak{p} -irreducible quotient, which can be realized as $H_0(\mathfrak{p}_+, \tilde{\mathbb{V}})$. Then it follows from the explicit descriptions via the Weyl group that $H_*(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V})$ consists of some of the irreducible components of $H_*(\mathfrak{q}_+, \tilde{\mathbb{V}})$. More generally, we prove that the Hasse diagram $W^{\mathfrak{q}}$ of \mathfrak{q} , which parametrizes the irreducible components in $H_*(\mathfrak{q}_+, \tilde{\mathbb{V}})$ can be written as a product $W_{\mathfrak{p}}^{\mathfrak{q}} \times W^{\mathfrak{p}}$, which leads to an isomorphism

$$H_*(\mathfrak{q}_+, \tilde{\mathbb{V}}) \cong H_*(\mathfrak{q}_+/\mathfrak{p}_+, H_*(\mathfrak{p}_+, \tilde{\mathbb{V}})),$$

see Theorem 3.3. The description as a product significantly simplifies the determination of the Hasse diagram for non-maximal parabolic subalgebras as we demonstrate in Example 3.2. Initially, the above isomorphism of homology groups is proved by comparing lowest weights. In preparation for the applications in [7], we conclude the paper by constructing an explicit isomorphism from \mathfrak{q} -invariant data. This is based on a filtration of $C_*(\mathfrak{q}_+, \tilde{\mathbb{V}})$ by \mathfrak{q} -submodules. While this does not lead to a filtered complex, it can be used to construct explicit \mathfrak{q} -equivariant maps, which realize the decomposition of $H_*(\mathfrak{q}_+, \tilde{\mathbb{V}})$ according to the degree in the second factor under the above isomorphism.

2. A RELATIVE VERSION OF KOSTANT'S THEOREM

Given two nested parabolic subalgebras $\mathfrak{q} \subset \mathfrak{p}$ in a (real or complex) semisimple Lie algebra \mathfrak{g} , we first develop a relative version of Kostant's Hodge theory. As remarked in the introduction, we work in a \mathfrak{q} -invariant setting as much as possible. Given a completely reducible representation \mathbb{V} of \mathfrak{p} , the main object of study thus is the standard complex for Lie algebra homology, while the other ingredients for the Hodge theory are of auxiliary nature. Then we compute the action of the algebraic Laplacian on an isotypical component. In the complex case, this leads to a description of Lie algebra homology in terms of a relative analog of the Hasse diagram.

2.1. Nested parabolic subalgebras. Throughout this article, we consider a (real or complex) semisimple Lie algebra \mathfrak{g} endowed with two nested parabolic subalgebras $\mathfrak{q} \subset \mathfrak{p} \subset \mathfrak{g}$. (In view of the applications we have in mind, this notation, as well as much of the notation in what follows is chosen in accordance with the literature on parabolic geometries.) It is well known that both in the real and in the complex case, parabolic subalgebras can be described in terms of so-called $|k|$ -gradings of \mathfrak{g} (for various values of $k \in \mathbb{N}$). Such a grading is a decomposition of \mathfrak{g} as a direct sum

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$$

such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ (with $\mathfrak{g}_{\ell} = \{0\}$ for $|\ell| > k$). Moreover, we make the technical assumptions that no simple ideal of \mathfrak{g} is contained in \mathfrak{g}_0 (which is a subalgebra of \mathfrak{g} by the grading property) and that the positive (negative) part of

the grading is generated as a Lie algebra by \mathfrak{g}_1 (\mathfrak{g}_{-1}). The parabolic subalgebra determined by such a grading is the non-negative part $\mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$ of the grading. It then turns out that the positive part is the nilradical of the parabolic subalgebra, while \mathfrak{g}_0 is its Levi factor.

Since we have to deal with two parabolics at the same time, we will avoid the use of explicit indices for the grading components. Following [4], we denote the decompositions of \mathfrak{g} determined by the two parabolic subalgebras by

$$\mathfrak{g} = \mathfrak{q}_- \oplus \mathfrak{q}_0 \oplus \mathfrak{q}_+ \quad \text{and} \quad \mathfrak{g} = \mathfrak{p}_- \oplus \mathfrak{p}_0 \oplus \mathfrak{p}_+,$$

respectively. For a parabolic subalgebra, the nilradical coincides with the annihilator with respect to the Killing form. Thus $\mathfrak{q} \subset \mathfrak{p}$ implies $\mathfrak{p}_+ \subset \mathfrak{q}_+$ and by symmetry of the grading we conclude that $\mathfrak{p}_- \subset \mathfrak{q}_-$. Since \mathfrak{p}_+ is an ideal in \mathfrak{p} , it is also an ideal in \mathfrak{q} and in \mathfrak{q}_+ .

The classification of parabolic subalgebras is done via structure theory. Given a $|k|$ -grading of a complex semisimple Lie algebra \mathfrak{g} , one shows that one can choose a Cartan subalgebra \mathfrak{h} contained in \mathfrak{g}_0 , so its adjoint action preserves the decomposition defined by the grading. The assumptions then easily imply that the root spaces corresponding to simple roots are either contained in \mathfrak{g}_0 or in \mathfrak{g}_1 . Denoting by $\Sigma \subset \Delta^0$ the set of those simple roots with root spaces contained in \mathfrak{g}_1 , it turns out that the grading is given by the Σ -height. This means that to determine the grading component containing a root space \mathfrak{g}_α , one expresses α as a linear combination of simple roots and then adds up the coefficients of the roots contained in Σ . This leads to a bijective correspondence between the set of conjugacy classes of parabolic subalgebras and subsets of Δ^0 , with the empty set corresponding to \mathfrak{g} and the full set Δ^0 corresponding to the Borel subalgebras. In the real case, one obtains a similar description in terms of restricted roots for a maximally non-compact Cartan subalgebra, see section 3.2.9 of [5].

The grading components for a $|k|$ -grading can be realized as the eigenspaces for the adjoint action of an element in the center of \mathfrak{g}_0 , the so-called grading element. In terms of structure theory, this element is always contained in the Cartan subalgebra. In particular, the grading elements associated to two nested parabolics commute, so the gradings are compatible. In particular, this implies that \mathfrak{p}_0 is invariant under the adjoint action of the grading element E_q for \mathfrak{q} . Hence we obtain the finer decomposition

$$(2.1) \quad \mathfrak{g} = \mathfrak{p}_- \oplus (\mathfrak{p}_0 \cap \mathfrak{q}_-) \oplus \mathfrak{q}_0 \oplus (\mathfrak{p}_0 \cap \mathfrak{q}_+) \oplus \mathfrak{p}_+$$

of \mathfrak{g} into a direct sum of subalgebras with the first two summands adding up to \mathfrak{q}_- and the last two summands adding up to \mathfrak{q}_+ .

Let us finally remark that it is no problem to choose parabolic subgroups corresponding to two nested parabolic subalgebras in a nice way. Given a Lie group G with Lie algebra \mathfrak{g} , we can first choose a parabolic subgroup $P \subset G$ corresponding to $\mathfrak{p} \subset \mathfrak{g}$. This means that P is a subgroup of the normalizer $N_G(\mathfrak{p})$ which contains the connected component of the identity of this normalizer. Then we consider the normalizer $N_P(\mathfrak{q})$ of \mathfrak{q} in P and choose a subgroup Q lying between this normalizer and its connected component of the identity, thus obtaining groups $Q \subset P \subset G$ corresponding to $\mathfrak{q} \subset \mathfrak{p} \subset \mathfrak{g}$.

2.2. Lie algebra homology. While the general theory of finite dimensional representations of a parabolic subalgebra is rather intractable, irreducible representations (and hence also completely reducible ones) are rather easy to understand. If $\mathfrak{p} \subset \mathfrak{g}$ is a parabolic subalgebra and \mathbb{V} is an irreducible representation of \mathfrak{p} , then the nilradical \mathfrak{p}_+ acts trivially on \mathbb{V} , so the representation descends to the reductive quotient $\mathfrak{p}/\mathfrak{p}_+ \cong \mathfrak{p}_0$. Conversely, a representation of \mathfrak{p}_0 is completely reducible, provided that the center acts diagonalizably, and one can extend such a representation to \mathfrak{p} (with \mathfrak{p}_+ acting trivially).

In the setting of two nested parabolic subalgebras $\mathfrak{q} \subset \mathfrak{p}$ in \mathfrak{g} and a completely reducible representation \mathbb{V} of \mathfrak{p} , we can first restrict the representation to the (nilpotent) subalgebra $\mathfrak{q}_+ \subset \mathfrak{p}$. As we have noted in Section 2.1, \mathfrak{q}_+ contains the nilradical \mathfrak{p}_+ of \mathfrak{p} , and clearly \mathfrak{p}_+ is an ideal in \mathfrak{q}_+ . Hence \mathbb{V} is automatically a representation of the Lie algebra $\mathfrak{q}_+/\mathfrak{p}_+$. Thus we can consider the standard complex for computing Lie algebra homology of $\mathfrak{q}_+/\mathfrak{p}_+$ with coefficients in \mathbb{V} . The spaces in this complex are

$$C_k(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V}) := \Lambda^k(\mathfrak{q}_+/\mathfrak{p}_+) \otimes \mathbb{V}.$$

Following [9], where the differential in this complex was obtained as the adjoint of a Lie algebra cohomology differential, we denote it by

$$\partial_\rho^* : \Lambda^k(\mathfrak{q}_+/\mathfrak{p}_+) \otimes \mathbb{V} \rightarrow \Lambda^{k-1}(\mathfrak{q}_+/\mathfrak{p}_+) \otimes \mathbb{V},$$

and call it the *relative Kostant–codifferential*. Explicitly, this differential is given by

$$(2.2) \quad \begin{aligned} \partial_\rho^*(Z_1 \wedge \cdots \wedge Z_k \otimes v) &:= \sum_i (-1)^i Z_1 \wedge \cdots \wedge \widehat{Z}_i \wedge \cdots \wedge Z_k \otimes Z_i \cdot v \\ &+ \sum_{i < j} (-1)^{i+j} [Z_i, Z_j] \wedge Z_1 \wedge \cdots \wedge \widehat{Z}_i \wedge \cdots \wedge \widehat{Z}_j \wedge \cdots \wedge Z_k \otimes v, \end{aligned}$$

for $Z_1, \dots, Z_k \in \mathfrak{q}_+/\mathfrak{p}_+$ and $v \in \mathbb{V}$, with hats denoting omission. It satisfies $\partial_\rho^* \circ \partial_\rho^* = 0$ and the homology groups of $(C_*(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V}), \partial_\rho^*)$ are the Lie algebra homology groups $H_*(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V})$.

Proposition 2.1. *Let $\mathfrak{q} \subset \mathfrak{p} \subset \mathfrak{g}$ be nested parabolic subalgebras, $Q \subset P \subset G$ be corresponding groups as in Section 2.1, and let \mathbb{V} be a finite dimensional, completely reducible representation of \mathfrak{p} .*

Then the spaces $C_(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V})$ are naturally \mathfrak{q} -modules such that the differentials ∂_ρ^* are \mathfrak{q} -equivariant. Moreover, this action has the property that $\mathfrak{q}_+ \cdot \ker(\partial_\rho^*) \subset \text{im}(\partial_\rho^*)$, so the induced representations of \mathfrak{q} on the homology groups $H_*(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V})$ are completely reducible. Finally, if \mathbb{V} is a completely reducible representation of the group P , then the module structures lift to Q and the differentials are Q -equivariant.*

Proof. As we have noted in Section 2.1, the adjoint action of Q preserves both \mathfrak{q}_+ and \mathfrak{p}_+ , so there is a natural induced action (by automorphisms) on $\mathfrak{q}_+/\mathfrak{p}_+$. Correspondingly, \mathfrak{q} acts on $\mathfrak{q}_+/\mathfrak{p}_+$ by derivations. On the other hand, \mathfrak{q} acts on \mathbb{V} by restriction of the \mathfrak{p} -action, and an action of P on \mathbb{V} can be restricted to the subgroup Q . Hence we obtain the claimed module structures on the spaces $C_*(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V})$. Of course the action $(\mathfrak{q}_+/\mathfrak{p}_+) \times \mathbb{V} \rightarrow \mathbb{V}$ is equivariant for the natural

\mathfrak{q} -action and for the Q -action in case that \mathbb{V} is a P -module. Together with the above this implies equivariancy of the differentials ∂_ρ^* by their definition.

Starting from the definition of ∂_ρ^* , a simple computation (see the proof of Lemma 3.3.2 in [5]) shows that for $\varphi \in \Lambda^k(\mathfrak{q}_+/\mathfrak{p}_+) \otimes \mathbb{V}$ and $Z \in \mathfrak{q}_+$ one gets

$$Z \cdot \varphi = -\partial_\rho^*((Z + \mathfrak{p}_+) \wedge \varphi) - (Z + \mathfrak{p}_+) \wedge \partial_\rho^*(\varphi).$$

This immediately implies that $\mathfrak{q}_+ \cdot \ker(\partial_\rho^*) \subset \text{im}(\partial_\rho^*)$, so \mathfrak{q}_+ acts trivially on the homology groups. Since by assumption the center of \mathfrak{q}_0 acts diagonalizably on all the involved representations, complete reducibility of the homology representations follows. \square

2.3. Lie algebra cohomology. The first step towards the computation of the Lie algebra homology groups $H_*(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V})$ is the construction of an adjoint to the Lie algebra homology differential. However, such an adjoint cannot be constructed as a \mathfrak{q} -equivariant map, one only obtains equivariancy under \mathfrak{q}_0 .

The Killing form B on \mathfrak{g} is compatible with the $|k|$ -grading determined by any parabolic subalgebra in the sense that it induces a duality (of \mathfrak{g}_0 -modules) between \mathfrak{g}_i and \mathfrak{g}_{-i} for all $i \neq 0$ and its restriction to \mathfrak{g}_0 is non-degenerate. For the two decompositions we are dealing with, this can be interpreted as an identification of \mathfrak{q}_+ with \mathfrak{q}_-^* while $\mathfrak{p}_+ \subset \mathfrak{q}_+$ can be identified with the annihilator of $\mathfrak{p} \cap \mathfrak{q}_- = \mathfrak{p}_0 \cap \mathfrak{q}_-$. Hence we can identify $\mathfrak{q}_+/\mathfrak{p}_+$, as a \mathfrak{q}_0 -module, with the dual of $\mathfrak{p}_0 \cap \mathfrak{q}_- \subset \mathfrak{g}$. Since both \mathfrak{p}_0 and \mathfrak{q}_- are Lie subalgebras of \mathfrak{g} and \mathfrak{q}_- is nilpotent, $\mathfrak{p}_0 \cap \mathfrak{q}_-$ is a nilpotent Lie subalgebra of \mathfrak{g} , which naturally acts on \mathbb{V} by the restriction of the \mathfrak{p}_0 -action.

Hence, for each k , we can identify the chain group $C_k(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V})$ as a \mathfrak{q}_0 -module with the cochain group $C^k(\mathfrak{q}_- \cap \mathfrak{p}_0, \mathbb{V})$. Consequently, we obtain the Lie algebra cohomology differential, which for consistency we denote by

$$\partial_\rho : C_k(\mathfrak{p}_+/\mathfrak{q}_+, \mathbb{V}) \rightarrow C_{k+1}(\mathfrak{p}_+/\mathfrak{q}_+, \mathbb{V}).$$

Viewing elements of $C_*(\mathfrak{p}/\mathfrak{q}, \mathbb{V})$ as alternating multilinear maps from $\mathfrak{q}_- \cap \mathfrak{p}_0$ to \mathbb{V} , this differential is given by

$$\begin{aligned} \partial_\rho \varphi(X_0, \dots, X_k) &:= \sum_{i=0}^k (-1)^i X^i \cdot \varphi(X_0, \dots, \widehat{X}_i, \dots, X_k) \\ &\quad + \sum_{i < j} (-1)^{i+j} \varphi([X_i, X_j], X_0, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_k), \end{aligned}$$

for $X_0, \dots, X_k \in \mathfrak{q}_- \cap \mathfrak{p}_0$ with hats denoting omission.

2.4. Algebraic Hodge decomposition. The first key step toward the proof of Kostant's theorem and its relative analog is that the two differentials ∂_ρ^* and ∂_ρ satisfy a property called disjointness by Kostant. This easily implies that they give rise to a Hodge-decomposition of the chain groups. To formulate this, we first define the obvious analog $\square_\rho = \partial_\rho^* \circ \partial_\rho + \partial_\rho \circ \partial_\rho^*$ of the Kostant Laplacian, which maps each $C_k(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V})$ to itself.

Lemma 2.2 (Hodge decomposition). *For any completely reducible representation \mathbb{V} of \mathfrak{p} and each $k = 0, \dots, \dim(\mathfrak{q}_+/\mathfrak{p}_+)$ one has a decomposition*

$$C_k(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V}) = \text{im}(\partial_\rho^*) \oplus \ker(\square_\rho) \oplus \text{im}(\partial_\rho)$$

as a direct sum of \mathfrak{q}_0 -modules. Moreover, the first two summands add up to $\ker(\partial_\rho^*)$, while the last two summands add up to $\ker(\partial_\rho)$. Consequently, both the Lie algebra homology group $H_k(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V})$ and the Lie algebra cohomology group $H^k(\mathfrak{q}_- \cap \mathfrak{p}_0, \mathbb{V})$ are isomorphic to $\ker(\square_\rho) \subset C_k(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V})$ as \mathfrak{q}_0 -modules.

Proof. The main step in the proof is to verify disjointness of the operators ∂_ρ and ∂_ρ^* in the sense of Kostant, i.e. that $\ker(\partial_\rho^*) \cap \text{im}(\partial_\rho) = \{0\}$ and $\ker(\partial_\rho) \cap \text{im}(\partial_\rho^*) = \{0\}$. Having verified this, the argument in the proof of Theorem 3.3.1 of [5] can be applied without changes to prove the Hodge decomposition.

Decomposing \mathbb{V} into a direct sum of irreducibles, we get an induced decomposition of all chain spaces, which is preserved by both operators, so it suffices to prove disjointness in the case that \mathbb{V} is irreducible. Moreover, disjointness of two maps clearly follows from disjointness of complex linear extensions. Using complexifications, we may thus without loss of generality assume that \mathfrak{g} is a complex semisimple Lie algebra, $\mathfrak{q} \subset \mathfrak{p} \subset \mathfrak{g}$ are complex parabolic subalgebras, and that \mathbb{V} is a complex irreducible representation of \mathfrak{p} . This means that \mathbb{V} is an irreducible representation of the reductive Lie algebra \mathfrak{p}_0 (extended trivially on \mathfrak{p}_+), so in particular the center $\mathfrak{z}(\mathfrak{p}_0)$ acts by scalars determined by a complex linear functional $\lambda : \mathfrak{z}(\mathfrak{p}_0) \rightarrow \mathbb{C}$.

We may further assume that we have chosen a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and a set of positive roots such that \mathfrak{p} and \mathfrak{q} are standard parabolics with respect to these choices, so they both contain \mathfrak{h} and all positive root spaces. Now we verify disjointness by constructing a positive definite inner product on the chain spaces for which the operators ∂_ρ^* and ∂_ρ are adjoint. To do this, we first consider the Cartan involution θ for \mathfrak{g} coming from the standard construction of a compact real form $\mathfrak{u} \subset \mathfrak{g}$ as in Proposition 2.3.1 of [5]. This acts as minus the identity on the real subspace $\mathfrak{h}_0 \subset \mathfrak{h}$ on which all roots have real values, so that for the grading elements we get $\theta(E_p) = -E_p$ and $\theta(E_q) = -E_q$. In particular, θ respects both \mathfrak{p}_0 and \mathfrak{q}_0 and exchanges $\mathfrak{p}_0 \cap \mathfrak{q}_-$ and $\mathfrak{p}_0 \cap \mathfrak{q}_+$. Moreover, the Killing form B of \mathfrak{g} is non-degenerate on $(\mathfrak{p}_0 \cap \mathfrak{q}_-) \oplus (\mathfrak{p}_0 \cap \mathfrak{q}_+)$ which shows that $B_\theta(X, Y) := -B(X, \theta(Y))$ restricts to a positive definite inner product on $\mathfrak{p}_0 \cap \mathfrak{q}_+$. Hence we also get an induced inner product on $\Lambda^*(\mathfrak{p}_0 \cap \mathfrak{q}_+)$.

As an automorphism of the Lie algebra \mathfrak{p}_0 , θ also respects the decomposition $\mathfrak{p}_0 = \mathfrak{z}(\mathfrak{p}_0) \oplus [\mathfrak{p}_0, \mathfrak{p}_0]$, and we denote the second summand by \mathfrak{p}_0^s . The fixed point set of $\theta|_{\mathfrak{p}_0^s}$ is $\mathfrak{u} \cap \mathfrak{p}_0^s$ so this is a compact real form of \mathfrak{p}_0^s . In case that the functional λ is non-zero, we next have to modify the restriction of θ to $\mathfrak{z}(\mathfrak{p}_0)$ in such a way that $\lambda \circ \theta = -\bar{\lambda}$, for example by choosing an isomorphism with \mathbb{C}^k with first coordinate λ and then pulling back complex conjugation.

Compactness of $\mathfrak{u} \cap \mathfrak{p}_0^s$ then implies that there is a Hermitian inner product $\langle \cdot, \cdot \rangle$ on \mathbb{V} for which elements of this subalgebra act by skew Hermitian endomorphisms. Now by construction, $\langle A \cdot v_1, v_2 \rangle = -\langle v_1, \theta(A) \cdot v_2 \rangle$. Together with the inner product on $\Lambda^*(\mathfrak{p}_0 \cap \mathfrak{p}_+)$ from above, we get inner products on all chain spaces. Having these inner products at hand, the proof of adjointness of ∂_ρ and ∂_ρ^* works exactly as in Proposition 3.1.1 of [5], and disjointness follows. \square

2.5. A \mathfrak{q} -invariant interpretation. At this point, we make a short digression, which is mainly needed for the geometric applications in [7]. For these applications, we need interpretations of what we have done so far in terms of \mathfrak{q} -modules. As we have noted in Section 2.2, this is not a problem for the complex $(C_*(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V}), \partial_\rho^*)$, which consists of \mathfrak{q} -modules and \mathfrak{q} -equivariant maps.

However, the Lie subalgebra $\mathfrak{p}_0 \cap \mathfrak{q}_-$ used in the construction of ∂_ρ does not carry a natural \mathfrak{q} -module structure. Of course, one could define such a structure via the \mathfrak{q}_0 -equivariant isomorphism with $(\mathfrak{q}_+/\mathfrak{p}_+)^*$ from Section 2.3, but then the Lie bracket is not \mathfrak{q} -equivariant. This is reflected in the fact that, viewed as maps on the spaces $C_*(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V})$, the Lie algebra cohomology differentials ∂_ρ are only \mathfrak{q}_0 -equivariant and not \mathfrak{q} -equivariant.

To solve this problem, we first observe that on a completely reducible representation \mathbb{V} of \mathfrak{p} , one obtains a natural grading similar to the $|k|$ -grading on \mathfrak{g} . In the case that \mathbb{V} is a complex irreducible representation of \mathfrak{p} , we can view it as a representation of the reductive algebra \mathfrak{p}_0 . This means that elements of the Cartan subalgebra act diagonalizably on \mathbb{V} , so in particular this is true for the grading element E_q which lies in the center of \mathfrak{q}_0 . Hence we can decompose \mathbb{V} into eigenspaces for E_q , which all are \mathfrak{q}_0 -invariant by construction. With respect to the grading of $\mathfrak{g} = \bigoplus \mathfrak{g}_i$ induced by \mathfrak{q} , this clearly has the property that the action of \mathfrak{g}_j maps the eigenspace with eigenvalue a to the eigenspace for $a + j$ for any j . So we can view the eigenspace decomposition of \mathbb{V} as defining a grading $\mathbb{V} = \mathbb{V}_0 \oplus \dots \oplus \mathbb{V}_N$ for some $N \in \mathbb{N}$ such that $\mathfrak{g}_i \cdot \mathbb{V}_j \subset \mathbb{V}_{i+j}$. Via forming direct sums and complexifications this extends to general completely reducible representations of \mathfrak{p} .

Now we can combine this with the grading on $\mathfrak{q}_+/\mathfrak{p}_+$ induced by the grading of \mathfrak{g} coming from \mathfrak{q} , to obtain \mathfrak{q}_0 -invariant gradings on all the chain spaces $C_*(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V})$. The isomorphism $(\mathfrak{q}_+/\mathfrak{p}_+)^* \cong \mathfrak{p}_0 \cap \mathfrak{q}_-$ is compatible with the gradings and in the picture of multilinear maps from $\mathfrak{p}_0 \cap \mathfrak{q}_-$ to \mathbb{V} the grading on $C_*(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V})$ is given by the usual homogeneity of multilinear maps between graded vector spaces.

While these gradings are not \mathfrak{q} -invariant, the fact that $\mathfrak{q} = \bigoplus_{i \geq 0} \mathfrak{g}_i$ immediately implies that the filtrations by right ends induced by these gradings all are \mathfrak{q} -invariant. Denoting the grading components by $C_*(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V})_i$, the corresponding filtration is defined by

$$C_*(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V})^j := \bigoplus_{i \geq j} C_*(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V})_i.$$

Given a filtered \mathfrak{q} -module, we can pass to the associated graded module, which by definition is just the direct sum of the quotients of each filtration component by the next smaller one. From the construction, it is clear that as a \mathfrak{q}_0 -module, the associated graded $\text{gr}(C_*(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V}))$ is isomorphic to $C_*(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V})$. But since by construction \mathfrak{q}_+ maps any filtration component to the next smaller one, \mathfrak{q}_+ acts trivially on $\text{gr}(C_*(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V}))$, so this is a completely reducible representation of \mathfrak{q} .

In general, there is neither a natural map from a filtered module to its associated graded nor a natural map in the other direction. However, the grading we start with, defines such a mapping. Explicitly, given $\varphi \in C_*(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V})$, we can uniquely

write $\varphi = \sum \varphi_i$ with $\varphi_i \in C_*(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V})_i$. Denoting by

$$q_i : C_*(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V})^i \rightarrow \text{gr}_i(C_*(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V}))$$

the canonical projection, our isomorphism is defined by mapping φ to $\sum_i q_i(\varphi_i)$. In this way, we can interpret the maps ∂_ρ^* , ∂_ρ and \square_ρ as \mathfrak{q} -homomorphisms defined on $\text{gr}(C_*(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V}))$ and we then have the Hodge decomposition on this associated graded.

On the other hand, ∂_ρ^* can also be considered as a \mathfrak{q} -homomorphism defined on $C_*(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V})$ itself. From the explicit formula in Section 2.2 it follows immediately that ∂_ρ^* actually respects the grading on the chain space. This implies that for $\varphi \in C_*(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V})^j$ we obtain $\partial_\rho^* \varphi \in C_{*-1}(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V})^j$ and $q_j(\partial_\rho^* \varphi) = \partial_\rho^*(q_j(\varphi))$, where the ∂_ρ^* on the right hand side is the one on the associated graded. This justifies denoting both maps by the same symbol.

2.6. The action of \square_ρ . The next step towards a relative version of Kostant's theorem is an explicit description of the action of \square_ρ on $C_*(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V})$. As a Lie algebra and as a \mathfrak{q}_0 -module, we can identify $\mathfrak{q}_+/\mathfrak{p}_+$ with $\mathfrak{q}_+ \cap \mathfrak{p}_0$, compare with Section 2.5. Hence we get an isomorphism $C_k(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V}) \cong \Lambda^k(\mathfrak{q}_+ \cap \mathfrak{p}_0) \otimes \mathbb{V}$, and the inclusion of the subalgebra $\mathfrak{q}_+ \cap \mathfrak{p}_0$ into \mathfrak{p}_0 induces an inclusion

$$j : C_k(\mathfrak{q}_+ \cap \mathfrak{p}_0, \mathbb{V}) \rightarrow C_k(\mathfrak{p}_0, \mathbb{V}).$$

This is compatible with the Lie algebra homology differentials, which we therefore both denote by ∂^* .

The Killing form B of \mathfrak{g} has non-degenerate restrictions to \mathfrak{p}_0 and \mathfrak{q}_0 , so from the decomposition (2.1), we see that it induces an isomorphism $\mathfrak{p}_0 \cong \mathfrak{p}_0^*$ which restricts to a duality between $\mathfrak{q}_+ \cap \mathfrak{p}_0$ and $\mathfrak{q}_- \cap \mathfrak{p}_0$. This restriction is exactly the duality we used in Section 2.3 to define ∂_ρ and of course, the full duality can be used to define a Lie algebra cohomology differential $\partial_{\mathfrak{p}_0}$ on $C_*(\mathfrak{p}_0, \mathbb{V})$.

Having this at hand, the computations in sections 3.3.2 and 3.3.3 of [5] can be used without any change in our situation. To formulate the result, we observe that any element $X \in \mathfrak{p}_0$ naturally acts on \mathbb{V} . Likewise, X acts on \mathfrak{p}_0 by the adjoint action and this induces an action of $\Lambda^* \mathfrak{p}_0$. Together, these actions determine an action on $C_*(\mathfrak{p}_0, \mathbb{V})$, and we write \mathcal{L}_X for the action of X . We can decompose $\mathcal{L}_X = \mathcal{L}_X^{\mathfrak{p}_0} + \mathcal{L}_X^{\mathbb{V}}$ where in the first part X acts on $\Lambda^* \mathfrak{p}_0$ only, while in the second part it acts only on \mathbb{V} . Using this, we can formulate the result as follows.

Proposition 2.3. *Let $\{\xi_\ell\}$ be a basis for \mathfrak{p}_0 , which is the union of a basis of $\mathfrak{q}_- \cap \mathfrak{p}_0$ and of a basis of $\mathfrak{q}_+ \cap \mathfrak{p}_0$ and let $\{\eta_\ell\}$ be the dual basis with respect to B . Let $j : C_*(\mathfrak{q}_+ \cap \mathfrak{p}_0, \mathbb{V}) \rightarrow C_*(\mathfrak{p}_0, \mathbb{V})$ be the inclusion. Then we have*

$$j \circ \square_\rho = \frac{1}{2} \left(- \sum_\ell \mathcal{L}_{\eta_\ell}^{\mathbb{V}} \mathcal{L}_{\xi_\ell}^{\mathbb{V}} - \sum_{\ell: \xi_\ell \in \mathfrak{q}_-} \mathcal{L}_{\eta_\ell} \mathcal{L}_{\xi_\ell} + \sum_{\ell: \xi_\ell \in \mathfrak{q}_+} \mathcal{L}_{\eta_\ell} \mathcal{L}_{\xi_\ell} \right) \circ j.$$

2.7. The action on isotypical components. To continue, we restrict to the complex case, i.e. we assume that $\mathfrak{q} \subset \mathfrak{p} \subset \mathfrak{g}$ are nested parabolic subalgebras in a complex semisimple Lie algebra \mathfrak{g} and that \mathbb{V} is a complex irreducible representation of \mathfrak{p}_0 . This means that \mathbb{V} is a complex irreducible representation of the semisimple part \mathfrak{p}_0^s of \mathfrak{p}_0 on which the center $\mathfrak{z}(\mathfrak{p}_0)$ acts diagonalizably.

Usually, one describes such representations by highest weights, but in our setting it is better to use the negatives of lowest weights. Such a weight is a complex linear functional λ on the Cartan subalgebra \mathfrak{h} of \mathfrak{g} . (Recall that \mathfrak{h} naturally decomposes as the direct sum of the center $\mathfrak{z}(\mathfrak{p}_0)$ and a Cartan subalgebra for \mathfrak{p}_0^s .) The functionals occurring in this way are exactly the \mathfrak{p} -algebraically integral ones which in addition are \mathfrak{p} -dominant, which means that $\langle \lambda, \alpha \rangle$ is a non-negative integer for all positive roots α such that $\mathfrak{g}_\alpha \subset \mathfrak{p}_0$.

As we have noted already, we can view the chain spaces as $C_*(\mathfrak{p}_0 \cap \mathfrak{q}_+, V)$ and they are completely reducible \mathfrak{q}_0 -modules. Now irreducible representations of \mathfrak{q}_0 are also determined by the negatives of their lowest weights which again are linear functionals on \mathfrak{h} . Here, they have to be \mathfrak{q} -algebraically integral and \mathfrak{q} -dominant, so the condition that $\langle \lambda, \alpha \rangle$ is a non-negative integer is only required if $\mathfrak{g}_\alpha \subset \mathfrak{q}_0$. In particular, for a \mathfrak{q} -dominant integral weight ν , there is the \mathfrak{q}_0 -isotypical component $W^\nu \subset C_*(\mathfrak{p}_0 \cap \mathfrak{q}_+, V)$ of lowest weight $-\nu$, which is the \mathfrak{q}_0 -submodule generated by all \mathfrak{q}_0 -lowest weight vectors of that weight.

Now we can compute the action of \square_ρ on any isotypical component. To formulate the result, we denote by $\delta_{\mathfrak{p}}$ the *lowest form* of the semisimple Lie algebra \mathfrak{p}_0^s , i.e. half of the sum of its positive roots.

Corollary 2.4. *Let $\mathfrak{q} \subset \mathfrak{p}$ be nested standard parabolic subalgebras in a complex semisimple Lie algebra \mathfrak{g} and let \mathbb{V} be a complex irreducible representation of \mathfrak{p} with lowest weight $-\lambda \in \mathfrak{h}^*$. Let $W^\nu \subset C_*(\mathfrak{p}_0 \cap \mathfrak{q}_-, \mathbb{V})$ be the \mathfrak{q}_0 -isotypical component of lowest weight $-\nu$.*

Then \square_ρ acts on W^ν by multiplication by the scalar $\frac{1}{2}(\|\lambda + \delta_{\mathfrak{p}}\|^2 - \|\nu + \delta_{\mathfrak{p}}\|^2)$, where the norm is induced by the Killing form of \mathfrak{g} .

Proof. To prove the result, it suffices to show that \square_ρ acts on a lowest weight vector of weight $-\nu$ by multiplication with the scalar in question. We use the formula for \square_ρ from Proposition 2.3 with respect to appropriately chosen dual bases. Recall that the Cartan subalgebra \mathfrak{h} of \mathfrak{g} is contained in \mathfrak{p}_0 and that B is positive definite on the subspace $\mathfrak{h}_0 \subset \mathfrak{h}$ on which all roots (of \mathfrak{g}) are real. We start by choosing an orthonormal basis $\{H_1, \dots, H_r\}$ for \mathfrak{h}_0 which then is a complex basis for \mathfrak{h} . Next, let $\Delta^+(\mathfrak{p}_0)$ be the set of those positive roots α of \mathfrak{g} for which the root space \mathfrak{g}_α is contained in \mathfrak{p}_0 . For each $\alpha \in \Delta^+(\mathfrak{p}_0)$, we choose elements $E_\alpha \in \mathfrak{g}_\alpha$ and $F_\alpha \in \mathfrak{g}_{-\alpha}$ such that $B(E_\alpha, F_\alpha) = 1$. Then $\{E_\alpha, H_i, F_\alpha\}$ is a basis for \mathfrak{p}_0 whose dual basis with respect to B is given by $\{F_\alpha, H_i, E_\alpha\}$. Moreover, for each α we see that $[E_\alpha, F_\alpha]$ is dual to α with respect to B .

Having chosen these bases, one completes the proof as for Proposition 3.3.4 in [5], using the evident decomposition $\Delta^+(\mathfrak{p}_0) = \Delta^+(\mathfrak{q}_0) \sqcup \Delta^+(\mathfrak{q}_+ \cap \mathfrak{p}_0)$ according to location of the root spaces. \square

2.8. The relative Hasse diagram. The statement of Kostant's theorem is based on a subset in the Weyl group W of \mathfrak{g} , the so-called Hasse diagram associated to a parabolic subalgebra. We next introduce a relative version of this. Recall that for a parabolic subalgebra, one defines the Weyl group as the Weyl group of the semisimple part of a Levi factor, which is naturally a subgroup of W and thus

acts on \mathfrak{h}^* . In our situation of two nested parabolics $\mathfrak{q} \subset \mathfrak{p} \subset \mathfrak{g}$ we thus have $W_{\mathfrak{q}} \subset W_{\mathfrak{p}} \subset W$.

Recall further that denoting by Δ^+ the set of positive roots of \mathfrak{g} , one associates to $w \in W$ the subset $\Phi_w := \{\alpha \in \Delta^+ : w^{-1}(\alpha) \in -\Delta^+\} \subset \Delta^+$ which uniquely determines w . Using a notation based on the location of root spaces as in Section 2.7 we can write $\Delta^+ = \Delta^+(\mathfrak{p}_0) \sqcup \Delta^+(\mathfrak{p}_+)$ and we can further decompose $\Delta^+(\mathfrak{p}_0) = \Delta^+(\mathfrak{q}_0) \sqcup \Delta^+(\mathfrak{p}_0 \cap \mathfrak{q}_+)$. It is well known (see section 3.2.15 of [5]) that $w \in W_{\mathfrak{p}}$ if and only if $\Phi_w \subset \Delta^+(\mathfrak{p}_0)$ and likewise for $W_{\mathfrak{q}}$. The *Hasse diagram* $W^{\mathfrak{p}}$ associated to the parabolic subalgebra \mathfrak{p} is defined as the set of those $w \in W$ for which $\Phi_w \subset \Delta^+(\mathfrak{p}_+)$, and likewise for $W^{\mathfrak{q}}$.

Definition 2.5. For two nested parabolic subalgebras $\mathfrak{q} \subset \mathfrak{p}$ in a complex semisimple Lie algebra \mathfrak{g} , we define the *relative Hasse diagram* $W_{\mathfrak{p}}^{\mathfrak{q}} \subset W$ by

$$W_{\mathfrak{p}}^{\mathfrak{q}} := \{w \in W : \Phi_w \subset \Delta^+(\mathfrak{p}_0 \cap \mathfrak{q}_+)\}$$

From the above discussion, we conclude that $W_{\mathfrak{p}}^{\mathfrak{q}}$ coincides with the intersection $W^{\mathfrak{q}} \cap W_{\mathfrak{p}}$ of the Hasse diagram of \mathfrak{q} with the Weyl group of \mathfrak{p} . On the other hand, as a set $W_{\mathfrak{p}}^{\mathfrak{q}}$ can be identified with the Hasse diagram of the parabolic subalgebra $(\mathfrak{q} \cap \mathfrak{p}_0^s) \subset \mathfrak{p}_0^s$. The definition we have chosen has the advantage that $W_{\mathfrak{p}}^{\mathfrak{q}}$ naturally acts on all of \mathfrak{h}^* .

Relative Hasse diagrams can be determined in a similar way as usual ones and some of the basic properties carry over to the relative case.

Lemma 2.6. *Let $\mathfrak{q} \subset \mathfrak{p} \subset \mathfrak{g}$ be two nested parabolic subalgebras in a complex semisimple Lie algebra and let W be the Weyl group of \mathfrak{g} .*

(1) *Let $\delta_{\mathfrak{p}}^{\mathfrak{q}}$ denote the sum of all fundamental weights corresponding to simple roots contained in $\Delta^+(\mathfrak{p}_0 \cap \mathfrak{q}_+)$. Then the map $w \mapsto w^{-1}(\delta_{\mathfrak{p}}^{\mathfrak{q}})$ restricts to a bijection between $W_{\mathfrak{p}}^{\mathfrak{q}}$ and the orbit of $\delta_{\mathfrak{p}}^{\mathfrak{q}}$ under $W_{\mathfrak{p}}$.*

(2) *For any \mathfrak{p} -dominant weight λ and any element $w \in W_{\mathfrak{p}}^{\mathfrak{q}} \subset W$, the weight $w(\lambda)$ is \mathfrak{q} -dominant.*

Proof. (1) By construction, $\delta_{\mathfrak{p}}^{\mathfrak{q}}$ is orthogonal to each simple root $\alpha \in \Delta^+(\mathfrak{q}_0)$ and thus it is stabilized by the reflection corresponding to such a root. Since these reflections generate $W_{\mathfrak{q}}$, we see that $w(\delta_{\mathfrak{p}}^{\mathfrak{q}}) = \delta_{\mathfrak{p}}^{\mathfrak{q}}$ for any $w \in W_{\mathfrak{q}} \subset W_{\mathfrak{p}}$.

Conversely, if $w \in W$ satisfies $w(\delta_{\mathfrak{p}}^{\mathfrak{q}}) = \delta_{\mathfrak{p}}^{\mathfrak{q}}$, then for a root α , we have

$$\langle \delta_{\mathfrak{p}}^{\mathfrak{q}}, w^{-1}(\alpha) \rangle = \langle w(\delta_{\mathfrak{p}}^{\mathfrak{q}}), \alpha \rangle = \langle \delta_{\mathfrak{p}}^{\mathfrak{q}}, \alpha \rangle,$$

so this is positive for each $\alpha \in \Delta^+(\mathfrak{q}_+)$. But this implies that $\Phi_w \subset \Delta^+(\mathfrak{q}_0)$ and hence $w \in W_{\mathfrak{q}}$.

Applying Proposition 5.13 of [9] (Proposition 3.2.15 in [5]) to the parabolic subalgebra $\mathfrak{q} \cap \mathfrak{p}_0^s \subset \mathfrak{p}_0^s$, we conclude that any element $w \in W_{\mathfrak{p}}$ can be uniquely written as $w = w_1 w_2$ with $w_1 \in W_{\mathfrak{q}}$ and $w_2 \in W_{\mathfrak{p}}^{\mathfrak{q}}$. Hence $w^{-1}(\delta_{\mathfrak{p}}^{\mathfrak{q}}) = w_2^{-1}(\delta_{\mathfrak{p}}^{\mathfrak{q}})$, and we conclude that $w \mapsto w^{-1}(\delta_{\mathfrak{p}}^{\mathfrak{q}})$ defines a surjection from $W_{\mathfrak{p}}^{\mathfrak{q}}$ onto the $W_{\mathfrak{p}}$ -orbit of $\delta_{\mathfrak{p}}^{\mathfrak{q}}$. But if $w, \tilde{w} \in W_{\mathfrak{p}}^{\mathfrak{q}}$ satisfy $w^{-1}(\delta_{\mathfrak{p}}^{\mathfrak{q}}) = \tilde{w}^{-1}(\delta_{\mathfrak{p}}^{\mathfrak{q}})$, then $w\tilde{w}^{-1}$ fixes $\delta_{\mathfrak{p}}^{\mathfrak{q}}$ and hence lies in $W_{\mathfrak{q}}$. If w and \tilde{w} were different, then $w = (w\tilde{w}^{-1})\tilde{w}$ would contradict uniqueness of the product decomposition.

(2) For $\alpha \in \Delta^+(\mathfrak{q}_0)$ and $w \in W_{\mathfrak{p}}^{\mathfrak{q}}$, we by definition know that $w^{-1}(\alpha) \in \Delta^+(\mathfrak{p})$. Hence for a \mathfrak{p} -dominant weight λ we have

$$0 \leq \langle \lambda, w^{-1}(\alpha) \rangle = \langle w(\lambda), \alpha \rangle,$$

so $w(\lambda)$ is \mathfrak{q} -dominant. \square

We will give an example for determining $W_{\mathfrak{p}}^{\mathfrak{q}}$ in Example 3.2 below.

2.9. The relative version of Kostant's theorem. The last ingredient needed to formulate the first main result of this article is the affine action of the Weyl group on weights. For a weight λ and $w \in W$, we define $w \cdot \lambda = w(\lambda + \delta) - \delta$, where $\delta = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$ is the lowest form of the Lie algebra \mathfrak{g} . This has the nice property that $w \cdot 0 = w(\delta) - \delta = -\sum_{\alpha \in \Phi_w} \alpha$, see Proposition 3.2.14 in [5]. From the proof of this Proposition and the fact that an element $w \in W_{\mathfrak{p}}$ has to preserve the set $\Delta^+(\mathfrak{p}_+)$ one easily concludes that $w \cdot 0 = w(\delta_{\mathfrak{p}}) - \delta_{\mathfrak{p}}$ for $w \in W_{\mathfrak{p}}$. In particular, for $w \in W_{\mathfrak{p}}^{\mathfrak{q}}$, we obtain $w \cdot \lambda = w(\lambda + \delta_{\mathfrak{p}}) - \delta_{\mathfrak{p}}$. If λ is \mathfrak{p} -dominant, then for any $\alpha \in \Delta^+(\mathfrak{p}_0)$ we have $\langle \lambda + \delta_{\mathfrak{p}}, \alpha \rangle \geq 1$ and using this, one concludes as in the proof of Lemma 2.6 that $w \cdot \lambda$ is always a \mathfrak{q} -dominant weight. Using this, we can now formulate:

Theorem 2.7. *[Relative version of Kostant's theorem] Consider two nested complex parabolic subalgebras $\mathfrak{q} \subset \mathfrak{p}$ in a complex semisimple Lie algebra \mathfrak{g} . Then for a finite dimensional complex irreducible representation \mathbb{V} of \mathfrak{p} with lowest weight $-\lambda \in \mathfrak{h}^*$, the homology space $H_*(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V})$ is a completely reducible representation of \mathfrak{q}_0 with the following structure:*

For a \mathfrak{q} -dominant weight ν , the isotypical component $H_(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V})^{\nu}$ of lowest weight $-\nu$ is non-zero if and only if $\nu = w \cdot \lambda$ for some $w \in W_{\mathfrak{p}}^{\mathfrak{q}}$. If this is the case, then the isotypical component is irreducible and contained in $H_{\ell(w)}(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V})$, where $\ell(w)$ is the length of w . The weight $-w \cdot \lambda$ even occurs with multiplicity one in $\Lambda^*(\mathfrak{q}_+/\mathfrak{p}_+) \otimes \mathbb{V}$.*

Proof. We have already noted that the weight $\nu_w := w \cdot \lambda$ is \mathfrak{q} -dominant. We have also seen above that $\nu_w = w(\lambda) - \sum_{\alpha \in \Phi_w} \alpha$. Now since $-\lambda$ is the lowest weight of the irreducible representation \mathbb{V} of \mathfrak{p}_0 , also $-w(\lambda)$ is a weight of \mathbb{V} . On the other hand, since $w \in W_{\mathfrak{p}}^{\mathfrak{q}}$, we have $\Phi_w \subset \Delta^+(\mathfrak{p}_0 \cap \mathfrak{q}_+)$. This exactly means that each $\alpha \in \Phi_w$ is a weight of $\mathfrak{q}_+/\mathfrak{p}_+$ and thus $\sum_{\alpha \in \Phi_w} \alpha$ is a weight of $\Lambda^k(\mathfrak{q}_+/\mathfrak{p}_+)$, where $k = |\Phi_w|$. It is well known that $|\Phi_w|$ coincides with the length $\ell(w)$. Hence we have verified that $-\nu_w$ indeed is a weight of the \mathfrak{q}_0 -representation $\Lambda^{\ell(w)}(\mathfrak{q}_+/\mathfrak{p}_+) \otimes \mathbb{V}$. On the other hand, since $\nu_w + \delta_{\mathfrak{p}} = w(\lambda + \delta_{\mathfrak{p}})$ we see that $\|\nu_w + \delta_{\mathfrak{p}}\| = \|\lambda + \delta_{\mathfrak{p}}\|$. Hence by Corollary 2.4, if $-\nu_w$ actually is a lowest weight in the representation $\Lambda^{\ell(w)}(\mathfrak{q}_+/\mathfrak{p}_+) \otimes \mathbb{V}$ then its isotypical component will be contained in $\ker(\square_{\rho})$ which is isomorphic to the homology by Lemma 2.2.

The main step to complete the proof now is to derive an analog of Lemma 5.12 of [9] (Lemma 3.3.5 of [5]). This is rather straightforward along the lines of Cartier's simplified argument from [8], so we just sketch it. A weight $-\nu$ of $\Lambda^*(\mathfrak{q}_+/\mathfrak{p}_+) \otimes \mathbb{V}$ can be written as

$$(2.3) \quad -\mu + \sum_{\alpha \in \Psi} \alpha,$$

where $-\mu$ is a weight of \mathbb{V} and Ψ is some subset of $\Delta^+(\mathfrak{p}_0 \cap \mathfrak{q}_+)$. Moreover, the multiplicity of $-\nu$ as a weight coincides with the sum of the multiplicities of the weights $-\mu$ in \mathbb{V} over all decompositions of $-\nu$ as in (2.3).

Fixing a weight $-\nu$ decomposed in this way, there is an element $w \in W_{\mathfrak{p}}$ such that $w(-\nu - \delta_{\mathfrak{p}})$ is \mathfrak{p} -dominant, and of course

$$w(-\nu - \delta_{\mathfrak{p}}) = w(-\mu) - w(\delta_{\mathfrak{p}} - \sum_{\alpha \in \Psi} \alpha).$$

Now $w(-\mu)$ is a weight of \mathbb{V} and thus can be obtained from $-\lambda$ by adding a linear combination of simple roots from $\Delta^+(\mathfrak{p}_0)$ with non-negative integral coefficients. As in the proof of Lemma 3.3.5 of [5], one next shows that $w(\delta_{\mathfrak{p}} - \sum_{\alpha \in \Psi} \alpha)$ is obtained by subtracting from $\delta_{\mathfrak{p}}$ a linear combination of some simple roots from $\Delta^+(\mathfrak{p}_0)$ with non-negative integral coefficients. Altogether, we see that $\lambda + \delta_{\mathfrak{p}} = w(\nu + \delta_{\mathfrak{p}}) + \sum n_i \alpha_i$ for some non-negative integers n_i and simple roots $\alpha_i \in \Delta^+(\mathfrak{p}_0)$. Using that $w(\nu + \delta_{\mathfrak{p}})$ is \mathfrak{p} -dominant, this easily implies that

$$\|\lambda + \delta_{\mathfrak{p}}\| \geq \|w(\nu + \delta_{\mathfrak{p}})\| = \|\nu + \delta_{\mathfrak{p}}\|$$

with equality if and only if all n_i are zero. The latter condition means that $w(\mu) = \lambda$ and that $w(\delta_{\mathfrak{p}} - \sum_{\alpha \in \Psi} \alpha) = \delta_{\mathfrak{p}}$. Hence we obtain $\mu = w^{-1}(\lambda)$ and $w^{-1}(\delta_{\mathfrak{p}}) = \delta_{\mathfrak{p}} - \sum_{\alpha \in \Psi} \alpha$. The last condition implies that $\Psi = \Phi_{w^{-1}}$, which shows that $\nu = \nu_{w^{-1}}$, $w^{-1} \in W_{\mathfrak{p}}^{\mathfrak{q}}$, and that there is only one possible decomposition of ν as in (2.3).

Using this uniqueness, multiplicity one of the weight $-w^{-1}(\lambda)$ in \mathbb{V} implies that $\nu_{w^{-1}}$ has multiplicity one as a weight of $C_*(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V})$. Finally, since $\lambda + \delta_{\mathfrak{p}}$ lies in the interior of the dominant Weyl chamber for \mathfrak{p} , it follows that for $w \neq w' \in W_{\mathfrak{p}}^{\mathfrak{q}}$ we get $\nu_w \neq \nu_{w'}$.

In view of Corollary 2.4, we can complete the proof by showing that each $-\nu_w$ for $w \in W_{\mathfrak{p}}^{\mathfrak{q}}$ is actually a lowest weight of $C_*(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V})$. But this follows as for the absolute version of Kostant's theorem, by showing that for $\alpha \in \Delta^+(\mathfrak{q}_0)$ one has $\|\nu_w + \alpha + \delta_{\mathfrak{p}}\| > \|\lambda + \delta_{\mathfrak{p}}\|$, so $-\nu_w - \alpha$ cannot be a weight of $C_*(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V})$. \square

3. RELATIVE AND ABSOLUTE HOMOLOGY

We start by describing the relation between relative and absolute Hasse diagrams. Since irreducible representations of \mathfrak{p} can have singular infinitesimal character, relative homology groups realize parts of an affine Weyl-orbits in either regular or singular infinitesimal character. In the case of regular infinitesimal character, we show that each affine Weyl orbit decomposes into a disjoint union of sequences of relative homology groups, and show how to obtain the individual sequences in terms of \mathfrak{q} -invariant data.

3.1. Lie algebra homology and affine Weyl orbits. The reason for the importance of Kostant's theorem in the theory of parabolic geometries is its relation to infinitesimal character. Given a parabolic subalgebra \mathfrak{q} in a semisimple Lie algebra \mathfrak{g} and corresponding groups $Q \subset G$, there is an associated geometric structure. These so-called parabolic geometries can be uniformly described in terms of Cartan connections, see the exposition in [5]. The class of parabolic geometries contains important examples like conformal structures and CR-structures and has been intensively studied during the last years. Via a construction of associated

bundles, any representation of Q determines a natural vector bundle on any manifold endowed with a parabolic geometry of type (G, Q) . One of the difficult and important questions then is to describe differential operators acting on sections of such bundles, which are intrinsic to the geometry in question.

This has a close connection to representation theory. The *homogeneous model* of parabolic geometries of type (G, Q) is the homogeneous space G/Q . For this example, the natural vector bundles as described above are exactly the homogeneous vector bundles, and differential operators intrinsic to the geometry are exactly those which are intertwining operators for that natural G -representations on spaces of smooth sections of homogeneous vector bundles. Via a duality, intertwining operators which are differential operators are related to homomorphisms of induced modules, see Section 1.4.10 in [5].

If one considers homogeneous bundles associated to irreducible representations of Q , then the resulting induced modules are generalized Verma modules. These are modules having an infinitesimal character, and clearly a non-zero homomorphism between two such modules can only exist if their infinitesimal characters agree. By a classical theorem of Harish-Chandra, this is true if and only if the highest weights of the inducing representations can be obtained from each other by the affine action of an element of the Weyl group. (The duality mentioned above is the reason why in the context related to differential operators it is more natural to work with negatives of lowest weights rather than highest weights.) This shows that invariant differential operators for parabolic geometries are rather rare, which is one of the reasons why they are interesting.

While the affine action of the Weyl group is easy to understand in terms of weights, it is a priori not at all clear (in particular in a geometric picture) how the representations in an affine Weyl orbit (which form the candidates for inducing domains and targets for invariant differential operators) are related. For the further discussion, we have to distinguish between affine Weyl orbits of *regular* and of *singular* infinitesimal character. Regular orbits are those involving points in the interiors of Weyl chambers, while singular orbits are contained in walls.

In the case of regular infinitesimal character (and of integral weights) Kostant's theorem provides a satisfactory solution to this problem. Any regular orbit contains a weight which lies in the interior of the dominant Weyl chamber, so it is of the form $\lambda + \delta$ for a \mathfrak{g} -dominant weight λ . If, in addition, λ is integral, then there is a finite-dimensional irreducible representation \mathbb{V} of \mathfrak{g} corresponding to λ . Kostant's theorem then implies that the completely reducible representation $H_*(\mathfrak{q}_+, \mathbb{V})$ is the direct sum of one copy of each of the irreducible \mathfrak{q} -representations with highest weights contained in the affine Weyl-orbit of λ . The \mathfrak{q} -invariant description in terms of the standard complex computing Lie algebra homology can then be directly transferred to geometry and this is the basis for the construction of BGG-sequences in [6].

The relative Lie algebra homology groups studied in Section 2 can serve a similar purpose in some cases of singular infinitesimal character, see Example 3.2. On the

other hand, in regular infinitesimal character, one can nicely relate absolute homology groups to relative ones. In both situations this has important consequences for invariant differential operators, which are studied in [7].

3.2. The relation between absolute and relative Hasse diagrams. Let us return to our standard setting of two nested parabolic subalgebras $\mathfrak{q} \subset \mathfrak{p}$ in a complex semisimple Lie algebra \mathfrak{g} . Then we have the two subgroups $W_{\mathfrak{q}} \subset W_{\mathfrak{p}}$, the two (absolute) Hasse diagrams $W^{\mathfrak{q}} \supset W^{\mathfrak{p}}$, and the relative Hasse diagram $W_{\mathfrak{p}}^{\mathfrak{q}}$ from Definition 2.5. All these subsets admit a nice description in terms of the set $\Phi_w \subset \Delta^+$ associated to a Weyl group element $w \in W$, compare with Section 2.8. The basis for the further discussion is the following simple result.

Proposition 3.1. *Let $\mathfrak{q} \subset \mathfrak{p}$ be two nested parabolic subalgebras in a complex semisimple Lie algebra \mathfrak{g} . Then multiplication in the Weyl group W of \mathfrak{g} induces a bijection*

$$W_{\mathfrak{p}}^{\mathfrak{q}} \times W^{\mathfrak{p}} \rightarrow W^{\mathfrak{q}}.$$

Moreover, for $w_1 \in W_{\mathfrak{p}}^{\mathfrak{q}}$ and $w_2 \in W^{\mathfrak{p}}$, we get $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$.

Proof. Take $w_1 \in W_{\mathfrak{p}}^{\mathfrak{q}}$ and $w_2 \in W^{\mathfrak{p}}$, so we know that $\Phi_{w_1} \subset \Delta^+(\mathfrak{p}_0 \cap \mathfrak{q}_+)$ and $\Phi_{w_2} \subset \Delta^+(\mathfrak{q}_+)$, and put $w = w_1 w_2$. For a positive root $\alpha \in \Delta^+(\mathfrak{q}_0)$, we thus get $w_1^{-1}(\alpha) \in \Delta^+$. Since $\mathfrak{q}_0 \subset \mathfrak{p}_0$ and $w_1 \in W_{\mathfrak{p}}$, we even get $w_1^{-1}(\alpha) \in \Delta^+(\mathfrak{p}_0)$. But this implies $w_2^{-1}(w_1^{-1}(\alpha)) \in \Delta^+$. Hence we see that $\Phi_w \subset \Delta^+(\mathfrak{q}_+)$, so $w \in W^{\mathfrak{q}}$.

Conversely, Proposition 5.13 of [9] (Proposition 3.2.15 in [5]) for the parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ says that each element $w \in W$ can be uniquely written as $w = w_1 w_2$ with $w_1 \in W_{\mathfrak{p}}$ and $w_2 \in W^{\mathfrak{p}}$ and that $\ell(w) = \ell(w_1) + \ell(w_2)$. Hence we can complete the proof by showing that $w \in W^{\mathfrak{q}}$ implies $w_1 \in W_{\mathfrak{p}}^{\mathfrak{q}}$. But this follows immediately from the proof of this result in [5], since there w_1 is obtained as the unique element of $W_{\mathfrak{p}}$ such that $\Phi_{w_1} = \Phi_w \cap \Delta^+(\mathfrak{p}_0) \subset \Delta^+(\mathfrak{p}_0 \cap \mathfrak{q}_+)$. \square

This result significantly simplifies the determination of the Hasse diagrams of non-maximal parabolics as well as the affine orbits of weights under this Hasse diagram. Let us describe this in a simple example, in which the Hasse diagram is available in Section 3.2.16 of [5].

Example 3.2. Consider $\mathfrak{g} = \mathfrak{sl}(4, \mathbb{C})$, let \mathfrak{p} be the maximal parabolic subalgebra corresponding to the first simple root and let \mathfrak{q} be the parabolic subalgebra corresponding to the first two simple roots. In the language of crossed Dynkin diagrams, these are

$$\mathfrak{p} = \times \text{---} \circ \text{---} \circ \quad \mathfrak{q} = \times \text{---} \times \text{---} \circ.$$

(1) Denoting the reflections corresponding to the simple roots by σ_i , $i = 1, 2, 3$ with roots numbered from left to right, it is clear by definition that $W_{\mathfrak{p}}$ is generated by σ_2 and σ_3 . Moreover, $\delta_{\mathfrak{p}}^{\mathfrak{q}}$ is just the second fundamental weight in this case. To represent a weight, we write it as a linear combination of fundamental weights and then write the coefficient over the vertex for the corresponding simple root in the crossed Dynkin diagram. In this language, it is easy to compute the action

of simple reflections on weights, see Section 3.2.16 of [5]. This shows that the $W_{\mathfrak{p}}$ -orbit of $\delta_{\mathfrak{p}}^{\mathfrak{q}}$ is

$$\begin{array}{c} 0 \quad 1 \quad 0 \\ \times \text{---} \circ \text{---} \circ \end{array} \rightarrow \begin{array}{c} 1 \quad -1 \quad 1 \\ \times \text{---} \circ \text{---} \circ \end{array} \rightarrow \begin{array}{c} 1 \quad 0 \quad -1 \\ \times \text{---} \circ \text{---} \circ \end{array}$$

and $W_{\mathfrak{p}}^{\mathfrak{q}} = \{e, \sigma_2, \sigma_2\sigma_3\}$ with the elements of length 0, 1, and 2. This immediately allows us to compute the affine $W_{\mathfrak{p}}^{\mathfrak{q}}$ -orbit of a general weight as

$$(3.1) \quad \begin{array}{c} a \quad b \quad c \\ \times \text{---} \times \text{---} \circ \end{array} \rightarrow \begin{array}{c} a+b+1 \quad -b-2 \quad b+c+1 \\ \times \text{---} \times \text{---} \circ \end{array} \rightarrow \begin{array}{c} a+b+c+2 \quad -b-c-3 \quad b \\ \times \text{---} \times \text{---} \circ \end{array}$$

Now the basic case of interest for computing homology is that the initial weight is \mathfrak{p} -dominant and integral, i.e. that a, b , and c are integers with $b, c \geq 0$. Then the three weights in the above pattern, which visibly are \mathfrak{q} -dominant and integral, are the negatives of the lowest weights of the irreducible representations $H_i(\mathfrak{q}/\mathfrak{p}_+, \mathbb{V})$ for $i = 0, 1, 2$. Here \mathbb{V} is the irreducible representation of \mathfrak{p} with the negative of the lowest weight equal to $\begin{array}{c} a \quad b \quad c \\ \times \text{---} \circ \text{---} \circ \end{array}$.

(2) Now we can use Proposition 3.1 to determine the Hasse diagram of \mathfrak{q} as well as affine Weyl orbits. Similarly as in (1), one can determine the Weyl orbit of $\begin{array}{c} 1 \quad 0 \quad 0 \\ \circ \text{---} \circ \text{---} \circ \end{array}$ to obtain that $W^{\mathfrak{p}} = \{e, \sigma_1, \sigma_1\sigma_2, \sigma_1\sigma_2\sigma_3\}$ with elements of length 0, 1, 2, and 3. Together with the description of $W_{\mathfrak{p}}^{\mathfrak{q}}$ from part (1), this immediately gives the 12-element set $W^{\mathfrak{q}}$. Likewise, we can easily determine the affine orbit of a general weight under $W^{\mathfrak{p}}$ as

$$(3.2) \quad \begin{array}{c} a \quad b \quad c \\ \times \text{---} \circ \text{---} \circ \end{array} \rightarrow \begin{array}{c} -a-2 \quad a+b+1 \quad c \\ \times \text{---} \times \text{---} \circ \end{array} \rightarrow \begin{array}{c} -a-b-3 \quad a \quad b+c+1 \\ \times \text{---} \circ \text{---} \circ \end{array} \rightarrow \begin{array}{c} -a-b-c-4 \quad a \quad b \\ \times \text{---} \circ \text{---} \circ \end{array}$$

The full $W^{\mathfrak{q}}$ -orbit is then obtained by shifting each of these weights according to (3.1).

In the case of integral weights of regular infinitesimal character, the initial weight in (3.2) is \mathfrak{g} -dominant and integral, and we see that the $W^{\mathfrak{q}}$ -orbit of the weight is the disjoint union of four $W_{\mathfrak{p}}^{\mathfrak{q}}$ -orbits as in (3.1) with the initial weights coming from (3.2).

The situation in singular infinitesimal character is more subtle. The largest singular orbits (for which each element lies in one wall but not in the intersection of two walls) are obtained by setting one coefficient in the initial weight equal to -1 . In each of the three possible cases, one verifies that the $W^{\mathfrak{p}}$ -orbit of the corresponding weight degenerates to a three-element set, with one of the elements \mathfrak{p} -dominant in each case. Using (3.1) to determine $W_{\mathfrak{p}}^{\mathfrak{q}}$ -orbits, we obtain three basic patterns in singular infinitesimal character, namely

$$\begin{array}{c} -1 \quad a \quad b \\ \times \text{---} \times \text{---} \circ \end{array} \rightarrow \begin{array}{c} a \quad -a-2 \quad a+b+1 \\ \times \text{---} \times \text{---} \circ \end{array} \rightarrow \begin{array}{c} a+b+1 \quad -a-b-3 \quad a \\ \times \text{---} \times \text{---} \circ \end{array}$$

$$\begin{array}{c} -a-2 \quad a \quad b \\ \times \text{---} \times \text{---} \circ \end{array} \rightarrow \begin{array}{c} -1 \quad -a-2 \quad a+b+1 \\ \times \text{---} \times \text{---} \circ \end{array} \rightarrow \begin{array}{c} b \quad -a-b-3 \quad a \\ \times \text{---} \times \text{---} \circ \end{array}$$

$$\begin{array}{c} -a-b-3 \quad a \quad b \\ \times \text{---} \times \text{---} \circ \end{array} \rightarrow \begin{array}{c} -b-2 \quad -a-2 \quad a+b+1 \\ \times \text{---} \times \text{---} \circ \end{array} \rightarrow \begin{array}{c} -1 \quad -a-b-3 \quad a \\ \times \text{---} \times \text{---} \circ \end{array}$$

with $a, b \geq 0$. For all these cases, the construction in [7] produces invariant differential operators in singular infinitesimal character.

To conclude this example, we remark that there is another maximal parabolic $\tilde{\mathfrak{p}}$ containing \mathfrak{q} , which corresponds to the crossed Dynkin diagram $\circ \text{---} \times \text{---} \circ$. For this case, the relative Hasse diagram $W_{\tilde{\mathfrak{p}}}^{\mathfrak{q}}$ is the two-element set $\{e, \sigma_1\}$ while $W^{\tilde{\mathfrak{p}}}$ has six elements, compare with Example 3.2.17 in [5]. Thus one obtains a decomposition

of $W^{\mathfrak{q}}$ into two copies of $W^{\tilde{\mathfrak{p}}}$. Both this decomposition and the one into three copies of $W^{\mathfrak{p}}$ can be spotted in the picture for $W^{\mathfrak{q}}$ on top of p. 330 of [5].

3.3. Decomposing absolute homology. From now on, we will restrict to the case of regular infinitesimal character and integral weights. This means that the affine Weyl orbit in question contains a \mathfrak{g} -dominant integral weight λ , so we can actually start with a finite dimensional complex irreducible representation \mathbb{V} of \mathfrak{g} with lowest weight $-\lambda$. By Kostant's theorem (applied to $\mathfrak{p} \subset \mathfrak{g}$), the \mathfrak{p} -dominant weights in the affine Weyl orbit of λ are exactly the highest weights of the \mathfrak{p} -irreducible summands in $H_*(\mathfrak{p}_+, \mathbb{V})$. If \mathbb{W} is one of these summands, then we can consider $H_*(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{W})$. If the lowest weight of the summand is $-w \cdot \lambda$ with $w \in W^{\mathfrak{p}}$, then we can apply Theorem 2.7 to see that this relative homology is the direct sum of one copy of each of the \mathfrak{q} -irreducible representations with negative of the lowest weight contained in the affine $W_{\mathfrak{p}}^{\mathfrak{q}}$ -orbit of $w \cdot \lambda$. In view of Proposition 3.1, this implies the following theorem in the complex case, the real case then follows by complexification.

Theorem 3.3. *Let $\mathfrak{q} \subset \mathfrak{p}$ be two nested parabolic subalgebras in a real or complex semisimple Lie algebra \mathfrak{g} and let \mathbb{V} be a completely reducible representation of \mathfrak{g} . Then, as a module over \mathfrak{q}_0 and for each $k = 0, \dots, \dim(\mathfrak{q}_+)$, one has*

$$H_k(\mathfrak{q}_+, \mathbb{V}) \cong \bigoplus_{i+j=k} H_i(\mathfrak{q}_+/\mathfrak{p}_+, H_j(\mathfrak{p}_+, \mathbb{V})).$$

As it stands, this is an abstract isomorphism deduced from coincidence of highest weights of irreducible components, and initially it is unclear how to obtain explicit maps realizing such an isomorphism. The rest of this article is devoted to giving a construction of an explicit \mathfrak{q} -invariant map inducing this isomorphism. This will be a crucial ingredient for the application to relative BGG sequences.

Observe that the statement of Theorem 3.3 actually looks like the result of a collapsed homology-version of a Hochschild-Serre spectral sequence (compare with Theorem 12.6 in [12]). Indeed, our description is based on a \mathfrak{q} -invariant filtration of the spaces $C_*(\mathfrak{q}_+, \mathbb{V})$ in the standard complex computing $H_*(\mathfrak{q}_+, \mathbb{V})$. However, this filtration is not compatible with the Lie algebra homology differential, which we will denote by $\partial_{\mathfrak{q}}^*$ in what follows, so we do not obtain a filtered complex. One could also involve a filtration on \mathbb{V} to make the standard complex into a filtered complex and probably use this to obtain an alternative proof of Theorem 3.3. However, the resulting filtration looks much less useful for the applications we have in mind.

There is a natural way to view the spaces $C_*(\mathfrak{q}_+, \mathbb{V})$ as k -linear alternating maps. Since the Killing form of \mathfrak{g} induces a \mathfrak{q} -equivariant duality between \mathfrak{q}_+ and $\mathfrak{g}/\mathfrak{q}$, we see that we can view $C_*(\mathfrak{q}_+, \mathbb{V})$ as $L(\Lambda^*(\mathfrak{g}/\mathfrak{q}), \mathbb{V})$. Further, since $\mathfrak{p} \subset \mathfrak{g}$ is a \mathfrak{q} -invariant subspace, so is $\mathfrak{p}/\mathfrak{q} \subset \mathfrak{g}/\mathfrak{q}$. Having this at hand, we can define our filtration.

Definition 3.4. Given nested parabolic subalgebras $\mathfrak{q} \subset \mathfrak{p} \subset \mathfrak{g}$, a completely reducible representation \mathbb{V} of \mathfrak{g} , and integers $0 < \ell \leq k$, we define subspaces $\mathcal{F}_k^{\ell} \subset C_k(\mathfrak{q}_+, \mathbb{V})$ as follows.

Viewing $\varphi \in C_k(\mathfrak{q}_+, \mathbb{V})$ as a k -linear, alternating map $(\mathfrak{g}/\mathfrak{q})^k \rightarrow \mathbb{V}$, we say that $\varphi \in \mathcal{F}_k^\ell$ if and only if it vanishes if at least $k - \ell + 1$ of its entries are from $\mathfrak{p}/\mathfrak{q} \subset \mathfrak{g}/\mathfrak{q}$. We further put $\mathcal{F}^\ell := \bigoplus_{k \geq \ell} \mathcal{F}_k^\ell$.

Lemma 3.5. *The subspaces $\mathcal{F}^\ell \subset C_*(\mathfrak{q}_+, \mathbb{V})$ define a \mathfrak{q} -invariant decreasing filtration, i.e. $\mathcal{F}^0 := C_*(\mathfrak{q}_+, \mathbb{V}) \supset \mathcal{F}^1 \supset \dots \supset \mathcal{F}^r \supset \mathcal{F}^{r+1} = \{0\}$, where $r = \dim(\mathfrak{g}/\mathfrak{p})$.*

Moreover, there is a \mathfrak{q} -equivariant surjection

$$\mathcal{F}^\ell \rightarrow \bigoplus_{k \geq \ell} L(\Lambda^{k-\ell}(\mathfrak{p}/\mathfrak{q}), \Lambda^\ell \mathfrak{p}_+ \otimes \mathbb{V})$$

with kernel $\mathcal{F}^{\ell+1}$.

Proof. The fact that each \mathcal{F}_k^ℓ is \mathfrak{q} -invariant follows immediately from the fact that $\mathfrak{p}/\mathfrak{q} \subset \mathfrak{g}/\mathfrak{q}$ is \mathfrak{q} -invariant. If $\ell + 1 \leq k$ and $\varphi \in \mathcal{F}_k^{\ell+1}$, then φ vanishes upon insertion of $k - \ell$ entries from $\mathfrak{p}/\mathfrak{q}$. But then evidently we have $\varphi \in \mathcal{F}_k^\ell$, so $\mathcal{F}^\ell \supset \mathcal{F}^{\ell+1}$.

Next, take $\varphi \in \mathcal{F}_k^\ell$ and view it as a k -linear alternating map $(\mathfrak{g}/\mathfrak{q})^k \rightarrow \mathbb{V}$. Inserting $k - \ell$ elements from $\mathfrak{p}/\mathfrak{q}$ into φ , one obtains an ℓ -linear alternating map $(\mathfrak{g}/\mathfrak{q})^\ell \rightarrow \mathbb{V}$. By assumption the resulting map vanishes upon insertion of a single element from $\mathfrak{p}/\mathfrak{q}$, so it descends to an element of $L(\Lambda^\ell(\mathfrak{g}/\mathfrak{p}), \mathbb{V}) \cong \Lambda^\ell \mathfrak{p}_+ \otimes \mathbb{V}$. Hence we have defined a map

$$(3.3) \quad \mathcal{F}_k^\ell \rightarrow L(\Lambda^{k-\ell}(\mathfrak{p}/\mathfrak{q}), \Lambda^\ell \mathfrak{p}_+ \otimes \mathbb{V}),$$

which is \mathfrak{q} -equivariant by construction, and whose kernel by definition coincides with $\mathcal{F}_k^{\ell+1}$.

If $\ell > r = \dim(\mathfrak{g}/\mathfrak{p}) = \dim(\mathfrak{p}_+)$, then the target space in (3.3) is trivial, so we see that $\mathcal{F}^\ell = \mathcal{F}^{\ell+1}$ if $\ell > r$. But since \mathcal{F}^ℓ evidently is zero for $\ell > \dim(\mathfrak{q}_+)$, we see that $\mathcal{F}^{r+1} = \{0\}$.

On the other hand, given an element of the target space in (3.3), we can first choose an arbitrary extension to a multilinear alternating map defined on $(\mathfrak{g}/\mathfrak{q})^{k-\ell}$. The values of this map can be interpreted as ℓ -linear maps $(\mathfrak{g}/\mathfrak{p})^\ell \rightarrow \mathbb{V}$. Since $\mathfrak{g}/\mathfrak{p}$ is a quotient of $\mathfrak{g}/\mathfrak{q}$, they can be viewed as ℓ -linear alternating maps defined on $(\mathfrak{g}/\mathfrak{q})^\ell$. Taking the arguments together and forming the complete alternation, we obtain an element $C_k(\mathfrak{q}_+, \mathbb{V})$ and a moment of thought shows that this is contained in \mathcal{F}_k^ℓ , so the map in (3.3) is surjective. \square

Next, we have the Lie algebra homology differential $\partial_{\mathfrak{p}}^* : \Lambda^\ell \mathfrak{p}_+ \otimes \mathbb{V} \rightarrow \Lambda^{\ell-1} \mathfrak{p}_+ \otimes \mathbb{V}$, which is \mathfrak{p} -equivariant and hence \mathfrak{q} -equivariant. In particular, the kernel $\ker(\partial_{\mathfrak{p}}^*)$ is a \mathfrak{q} -invariant subspace in $\Lambda^\ell \mathfrak{p}_+ \otimes \mathbb{V}$. The maps with values in this subspace form a \mathfrak{q} -invariant subspace in $L(\Lambda^{k-\ell}(\mathfrak{p}/\mathfrak{q}), \Lambda^\ell \mathfrak{p}_+ \otimes \mathbb{V})$, and we denote by $\tilde{\mathcal{F}}_k^\ell \subset \mathcal{F}_k^\ell$ the preimage of this subspace under the map from Lemma 3.5. Hence $\tilde{\mathcal{F}}_k^\ell \subset \mathcal{F}_k^\ell$ is a \mathfrak{q} -invariant subspace, which by Lemma 3.5 contains $\mathcal{F}_k^{\ell+1}$, and we define $\tilde{\mathcal{F}}^\ell = \bigoplus_{k \geq \ell} \tilde{\mathcal{F}}_k^\ell$.

Using the \mathfrak{p} -equivariant quotient map $\ker(\partial_{\mathfrak{p}}^*) \rightarrow H_\ell(\mathfrak{p}_+, \mathbb{V})$ and the map from Lemma 3.5, we obtain a \mathfrak{q} -equivariant surjection

$$(3.4) \quad \pi : \tilde{\mathcal{F}}_k^\ell \rightarrow L(\Lambda^{k-\ell}(\mathfrak{p}/\mathfrak{q}), H_\ell(\mathfrak{p}_+, \mathbb{V})) \cong \Lambda^{k-\ell}(\mathfrak{q}_+/\mathfrak{p}_+) \otimes H_\ell(\mathfrak{p}_+, \mathbb{V}),$$

which by construction vanishes on $\mathcal{F}_k^{\ell+1} \subset \tilde{\mathcal{F}}_k^\ell$.

3.4. Relating absolute and relative homology. As a next step, we can clarify the compatibility of the map π from (3.4) with the Lie algebra homology differential $\partial_q^* : \Lambda^k \mathfrak{q}_+ \otimes \mathbb{V} \rightarrow \Lambda^{k-1} \mathfrak{q}_+ \otimes \mathbb{V}$.

Proposition 3.6. (1) *For the Lie algebra homology differential ∂_q^* , it holds that $\ker(\partial_q^*) \cap \mathcal{F}^\ell \subset \tilde{\mathcal{F}}^\ell$, $\partial_q^*(\mathcal{F}^\ell) \subset \mathcal{F}^{\ell-1}$ and $\partial_q^*(\tilde{\mathcal{F}}^\ell) \subset \tilde{\mathcal{F}}^\ell$.*

(2) *Denoting by ∂_ρ^* the relative Lie algebra homology differential acting on the space $\Lambda^{k-\ell}(\mathfrak{q}_+/\mathfrak{p}_+) \otimes H_\ell(\mathfrak{p}_+, \mathbb{V})$, $\pi \circ \partial_q^*$ coincides with $\partial_\rho^* \circ \pi$ up to sign.*

Proof. To prove the statement, we use the decomposition $\mathfrak{q}_+ = (\mathfrak{q}_+ \cap \mathfrak{p}_0) \oplus \mathfrak{p}_+$ from Section 2.1, noting that the second summand is an ideal in \mathfrak{q}_+ , while the first summand is a Lie subalgebra isomorphic to $\mathfrak{q}_+/\mathfrak{p}_+$. This gives rise to a bigrading on $C_*(\mathfrak{q}_+, \mathbb{V})$ as

$$\Lambda^k \mathfrak{q}_+ \otimes \mathbb{V} = \bigoplus_{r+s=k} \Lambda^r(\mathfrak{q}_+ \cap \mathfrak{p}_0) \otimes \Lambda^s \mathfrak{p}_+ \otimes \mathbb{V} =: \bigoplus_{r+s=k} \Lambda^{(r,s)} \mathfrak{q}_+ \otimes \mathbb{V}.$$

Since \mathfrak{p}_+ is the annihilator of \mathfrak{p} in \mathfrak{q} , we see that $\mathcal{F}_k^\ell = \bigoplus_{s \geq \ell} \Lambda^{(k-s,s)} \mathfrak{q}_+ \otimes \mathbb{V}$ for $k \geq \ell$, and that the projection from Lemma 3.5 corresponds to extracting the component in $\Lambda^{(k-\ell,\ell)} \mathfrak{q}_+ \otimes \mathbb{V}$.

Now consider a decomposable element of $\Lambda^{(r,s)} \mathfrak{q}_+ \otimes \mathbb{V}$, which we denote as

$$(3.5) \quad Z_1 \wedge \cdots \wedge Z_r \wedge W_1 \wedge \cdots \wedge W_s \otimes v,$$

so the Z 's are in $\mathfrak{q}_+ \cap \mathfrak{p}_0$ and the W 's are in \mathfrak{p}_+ . Moreover, the bracket of two W 's and the bracket of a Z with a W lies in \mathfrak{p}_+ , while the bracket of two Z 's lies in $\mathfrak{q}_+ \cap \mathfrak{p}_0$.

Now the formula for the Lie algebra homology differential (compare with (2.2)) immediately implies that ∂_q^* maps this element to the sum of the components with bidegrees $(r-1, s)$ and $(r, s-1)$. On the one hand, this implies that $\partial_q^*(\mathcal{F}_k^\ell) \subset \mathcal{F}_{k-1}^{\ell-1}$.

On the other hand, we see that this construction makes $C_*(\mathfrak{q}_+, \mathbb{V})$ into a double complex, and we denote by ∂_1^* and ∂_2^* the two components of ∂_q^* . Then we obtain the usual relations $(\partial_i^*)^2 = 0$ for $i = 1, 2$ and $\partial_1^* \partial_2^* = -\partial_2^* \partial_1^*$. From the definition it also follows that the component of degree $(r, s-1)$ consists of the summands in which a W acts on v and the summands containing the brackets of two W 's. This easily implies that $\partial_2^* = (-1)^r \text{id} \otimes \partial_{\mathfrak{p}}^*$.

From the construction in Section 3.3 it is also clear that

$$\tilde{\mathcal{F}}_k^\ell \subset \mathcal{F}_k^\ell \cong \bigoplus_{\ell \leq s \leq k} \Lambda^{(k-s,s)} \mathfrak{q}_+ \otimes \mathbb{V}$$

exactly consists of those elements, for which the component in $\Lambda^{(k-\ell,\ell)} \mathfrak{q}_+ \otimes \mathbb{V}$ lies in the kernel of $\text{id} \otimes \partial_{\mathfrak{p}}^*$. This readily implies the first statement in (1) as well as $\partial_q^*(\tilde{\mathcal{F}}_k^\ell) \subset \mathcal{F}_{k-1}^\ell$. Moreover, an element $\varphi \in \tilde{\mathcal{F}}_k^\ell$ can be written as $\varphi = \varphi_1 + \varphi_2$ with $\partial_2^*(\varphi_1) = 0$ and $\varphi_2 \in \mathcal{F}_k^{\ell+1}$. But then $\partial_q^*(\varphi)$ is congruent to $\partial_1^*(\varphi_1) + \partial_2^*(\varphi_2)$ modulo $\mathcal{F}_{k-1}^{\ell+1}$. Since both these summands lie in $\ker(\partial_2^*)$ we conclude that $\partial_q^*(\varphi) \in \tilde{\mathcal{F}}_{k-1}^\ell$, which completes the proof of (1).

Returning to the decomposable element of $\Lambda^{(r,s)} \mathfrak{q}_+ \otimes \mathbb{V}$ from (3.5), we next observe that there is a natural action of the Lie algebra $\mathfrak{q}_+ \cap \mathfrak{p}_0$ on $\Lambda^s \mathfrak{p}_+ \otimes \mathbb{V}$,

which we denote by \bullet . Explicitly, $Z \bullet (W_1 \wedge \cdots \wedge W_j \otimes v)$ is given by

$$W_1 \wedge \cdots \wedge W_j \otimes Z \cdot v + \sum_i W_1 \wedge \cdots \wedge [Z, W_i] \wedge \cdots \wedge W_j \otimes v.$$

This easily implies that, in terms of this action, ∂_1^* maps the element (3.5) to

$$\begin{aligned} & \sum_{i=1}^r (-1)^i Z_1 \wedge \cdots \widehat{Z}_i \cdots \wedge Z_r \otimes Z_i \bullet (W_1 \wedge \cdots \wedge W_j \otimes v) \\ & + \sum_{i < j} (-1)^{i+j} [Z_i, Z_j] \wedge Z_1 \wedge \cdots \widehat{Z}_i \cdots \widehat{Z}_j \cdots \wedge Z_r \wedge W_1 \wedge \cdots \wedge W_s \otimes v. \end{aligned}$$

Clearly, $\ker(\partial_\rho^*) \subset \Lambda^s \mathfrak{p}_+ \otimes \mathbb{V}$ is invariant under the action \bullet , and projecting further to $H_s(\mathfrak{p}_+, \mathbb{V})$ the action \bullet corresponds to the natural action of $\mathfrak{q}_+/\mathfrak{p}_+$. The definition of ∂_ρ^* then implies the statement in (2). \square

Using this, we can now construct an explicit map relating absolute and relative homology groups. Consider the intersection $\ker(\partial_\mathfrak{q}^*) \cap \mathcal{F}_k^\ell$, which is a \mathfrak{q} -submodule in $C_k(\mathfrak{q}_+, \mathbb{V})$. By part (1) of Proposition 3.6, this is contained in $\widetilde{\mathcal{F}}_k^\ell$ so π is defined on this subspace. By part (2) of Proposition 3.6, this restriction has values in $\ker(\partial_\rho^*) \subset \Lambda^{k-\ell}(\mathfrak{q}_+/\mathfrak{p}_+) \otimes H_\ell(\mathfrak{p}_+, \mathbb{V})$. Hence we can postcompose with the canonical map to relative homology to obtain a \mathfrak{q} -equivariant map

$$(3.6) \quad \Pi : \ker(\partial_\mathfrak{q}^*) \cap \mathcal{F}_k^\ell \rightarrow H_{k-\ell}(\mathfrak{q}_+/\mathfrak{p}_+, H_\ell(\mathfrak{p}_+, \mathbb{V})).$$

Using this, we can formulate our final result.

Theorem 3.7. *For $\ell \leq k$, the map Π vanishes on $\text{im}(\partial_\mathfrak{q}^*) \cap \mathcal{F}_k^\ell$ and descends to a \mathfrak{q} -equivariant surjection*

$$H_k(\mathfrak{q}_+, \mathbb{V}) \rightarrow H_{k-\ell}(\mathfrak{q}_+/\mathfrak{p}_+, H_\ell(\mathfrak{p}_+, \mathbb{V})).$$

Proof. Via complexifications and direct sums, it suffices to prove this in the case that the Lie algebras are complex and that \mathbb{V} is a complex irreducible representation of \mathfrak{g} . Fix an irreducible component \mathbb{W} of $H_\ell(\mathfrak{p}_+, \mathbb{V})$. If $-\lambda$ is the lowest weight of \mathbb{V} , then the lowest weight of \mathbb{W} must be $-w_2 \cdot \lambda$ for an element $w_2 \in W^\mathfrak{p}$ of length ℓ . By Theorem 2.7, the \mathfrak{q} -representation $H_*(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{W})$ is completely reducible with each irreducible component occurring with multiplicity one. Moreover, the lowest weights of these components are exactly the weights $-w_1 \cdot w_2 \cdot \lambda$ for $w_1 \in W^\mathfrak{q}$.

From Proposition 3.1, we know that $w := w_1 w_2$ lies in $W^\mathfrak{q}$ and denoting by k its length, we see that w_1 has length $k - \ell$. By Kostant's theorem, $H_k(\mathfrak{q}_+, \mathbb{V})$ contains an irreducible representation of \mathfrak{q} with lowest weight $-w \cdot \lambda$, and we can consider the corresponding lowest weight vector $\varphi \in \Lambda^k \mathfrak{q}_+ \otimes \mathbb{V}$. This is the wedge product of root vectors $e_\alpha \in \mathfrak{g}_\alpha$ for each $\alpha \in \Phi_w$ tensorized with a certain weight vector of \mathbb{V} . Further, $\Phi_w \subset \Delta^+(\mathfrak{q}_+)$ and $\Phi_{w_1} = \Phi_w \cap \Delta^+(\mathfrak{p}_0)$, and this has $k - \ell$ elements. Hence in the language of the proof of Proposition 3.6, the lowest weight vector is contained in $\Lambda^{(k-\ell, \ell)} \mathfrak{q}_+ \otimes \mathbb{V}$, so it lies in \mathcal{F}_k^ℓ but not in $\mathcal{F}_k^{\ell+1}$. Of course, it also lies in $\ker(\partial_\mathfrak{q}^*)$.

Hence φ has non-trivial image in the quotient $\mathcal{F}_k^\ell / \mathcal{F}_k^{\ell+1}$ and its image under the map from (3.3) is a decomposable element of $\Lambda^{k-\ell}(\mathfrak{q}_+/\mathfrak{p}_+) \otimes \ker(\partial_\rho^*)$. The second component of this is a weight vector and by the multiplicity-one result in Kostant's theorem (applied to $H_*(\mathfrak{p}_+, \mathbb{V})$), our element has to have nontrivial

image in $\Lambda^{k-\ell}(\mathfrak{q}_+/\mathfrak{p}_+) \otimes H_\ell(\mathfrak{p}_+, \mathbb{V})$. More precisely, this image must be the tensor product of a decomposable element of $\Lambda^{k-\ell}(\mathfrak{q}_+/\mathfrak{p}_+)$ with the lowest weight vector for the irreducible component \mathbb{W} from above. By part (2) of Proposition 3.6 our element lies in $\ker(\partial_\rho^*)$ and by the multiplicity-one part of Theorem 2.7, it must be a harmonic element contained in the component in the homology corresponding to w_1 . This shows that $\Pi(\varphi)$ is a lowest weight vector for the irreducible component in question, so Π is surjective.

To complete the proof, it remains to show that Π vanishes on $\text{im}(\partial_q^*)$. But if Π would be non-zero on that subspace, then the image of $\Pi|_{\text{im}(\partial_q^*)}$ would contain one of the \mathfrak{q}_0 -irreducible components of $H_*(\mathfrak{q}_+/\mathfrak{p}_+, H_\ell(\mathfrak{p}_+, \mathbb{V}))$. But the lowest weights of each of these components is a lowest weight of an irreducible component of $H_*(\mathfrak{q}_+, \mathbb{V})$, so a weight vector of that weight cannot be contained in $\text{im}(\partial_q^*)$ by the multiplicity-one part of Kostant's theorem. \square

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